45 bodova

23.5 (2)

28 (3)

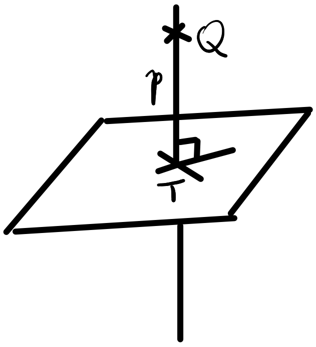
34 (4)

39.5 (5)

mat4 4. srpnja 2023. IME i PREZIME:

5 bodova

1. Promatraj ravninu N s jednadžbom $x+y+z+1=0$ i točku $Q(0,0,1)$ koja je van ravnine. Nadji pravac p koji prolazi kroz Q, a okomit je na N u parametarskom obliku i probodište tog pravca s N (dakle okomitu projekciju Q na N).



T – probodište

normalni vektor na $x+y+z+1=0$

p... $\vec{r}(t) = T + t\vec{n}$

$\vec{n}\,(1,1,1)$

Q je primjer, za neki t

$(0,0,1) = (x_T, y_T, z_T) + t\,(1,1,1)$

$\rightarrow x_T = -t,\ y_T = -t,\ z_T = 1-t$, $T \in N$

$(-t) + (-t) + (1-t) + 1 = 0$

$2 - 3t = 0 \rightarrow t = \frac{2}{3}$ za točku T

T je za $t=0$, dakle $T(x_T, y_T, z_T) = \left(-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

p... $\vec{r}(t) = \left(-\frac{2}{3} + t, -\frac{2}{3} + t, \frac{1}{3} + t\right) = T + t(1,1,1)$

4 boda

2. Promatrajmo 2×2 matrice s elementima u tijelu kvaterniona, gdje je $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$. Izračunaj

$$\begin{pmatrix} j & 2 \\ i & k \end{pmatrix} \cdot \begin{pmatrix} k+i & j \\ 1 & j \end{pmatrix} = \begin{pmatrix} j(k+i)+2\cdot 1 & j\cdot j + 2\cdot j \\ i(k+i)+k\cdot 1 & i\cdot j + k\cdot j \end{pmatrix}$$

$$= \begin{pmatrix} i-k+2 & -1+2j \\ -j-1+k & k-i \end{pmatrix} = \begin{pmatrix} 2+i-k & -1+2j \\ -1-j-k & -i+k \end{pmatrix}$$

3. Ispiši sve inverzije u permutaciji, napiši broj inverzija i paritet

4 boda

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 4 & 5 & 1 & 3 \end{pmatrix}$$

(1,2), (1,3), (1,4), (1,5), (1,6), (2,5), (3,5), (3,6), (4,5), (4,6)

10 inverzija

parna (+)

Inverzija permutacije s je par (i,j) gdje je i< j, a s(j)> s(i). Npr. (1,6) ide u (6,3) pa je inverzija, a (2,4) ide u (2,5) pa nije inverzija.

Paritet je paran ili +1 ako je broj inverzija paran.
Paritet je neparan ili -1 ako je broj inverzija neparan. **(Dakle (-1) potencirano na broj inverzija.)**

5 bodova

4. Promatramo trostranu piramidu kojoj su četiri vrha

$$E(2,1,0), F(-1,2,3), G(-2,4,1), H(0,0,-2).$$

Nadji volumen piramide.

$E(2,1,0)$

$F(-1,2,3)$

$G(-2,4,1)$

$H(0,0,-2)$

$$\begin{vmatrix} \vec{EF} \\ \vec{EG} \\ \vec{EH} \end{vmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ -4 & 3 & 1 \\ -2 & -1 & -2 \end{pmatrix}$$

$$V = \pm\frac{1}{12} \begin{vmatrix} -2 & 1 & 3 \\ -4 & 3 & 1 \\ -2 & -1 & -2 \end{vmatrix} =$$

$$\pm\frac{1}{6}\left(-2\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}\right)$$

$$= \pm\frac{1}{6}\left(-2(-5) + 4(1) - 2(-8)\right) = 5$$

$$10 + 4 + 16$$

$\{hg \mid h \in H\}$

2+2+2 5. a) Definiraj normalnu podgrupu.

Podgrupa H je normalna ako je svaka njena lijeva susjedna klasa H g (g proizvoljni ali fiksni element u G) jednaka desnoj susjednoj klasi g H. Alternativno, za svaki g $\in G$

$\{ghg^{-1} \mid h \in H\} = gHg^{-1} = H$

$\{g \cdot h \mid h \in H\}$

b) Napiši Cayleyev teorem u teoriji grupa..

Svaka grupa je izomorfna podgrupi permutacija nekog skupa.

c) Napiši Lagrangeov teorem u teoriji grupa.

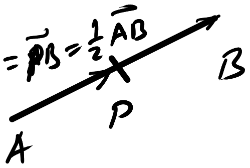
Broj elemenata u konačnoj grupi je uvijek djeljiv s brojem elemenata u ma kojoj njezinoj podgrupi.

4 boda 6. Trokut $\triangle ABG$ ima koordinate vrhova (u ravnini) $A(1,2,3)$, $B(0,1,4)$, $G(2,2,2)$. Nadji polovište stranice AB i koordinate težišta T tog trokuta.

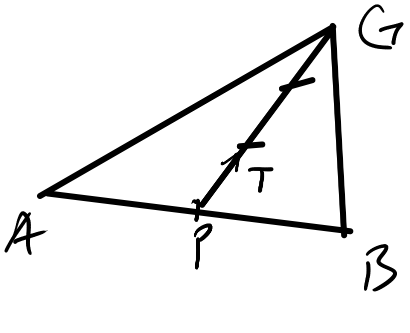
$\vec{AB}(-1,-1,1)$

$$P_{AB} = A + \frac{1}{2}\vec{AB} = (1,2,3) + \frac{1}{2}\underbrace{(0-1,\ 1-2,\ 4-3)}_{(0,1,4)-(1,2,3)} = \left(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}\right)$$

$\vec{AP} = \vec{PB} = \frac{1}{2}\vec{AB}$



ili $x_p = \frac{x_A + x_B}{2} = \frac{1+0}{2}$, $y_p = \frac{y_A + y_B}{2} = \frac{2+1}{2}$, $z_p = \frac{z_A + z_B}{2} = \frac{3+4}{2}$



$\vec{PT} = \frac{1}{3}\vec{PG}$ **po teoremu o težištu i težišnicama**

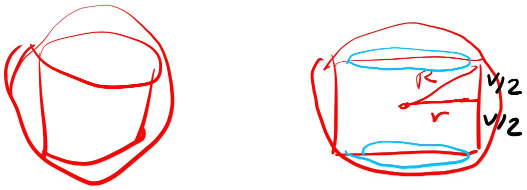
$$T = P + \frac{1}{3}\vec{PG} = \left(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}\right) + \frac{1}{3}\left(2-\frac{1}{2}, 2-\frac{3}{2}, 2-\frac{7}{2}\right) = \left(1, \frac{5}{3}, 3\right)$$

alternativno, $x_T = \frac{x_A + x_B + x_G}{3} = \frac{1+0+2}{3} = 1$

$$y_T = \frac{y_A + y_B + y_G}{3} = \frac{2+1+2}{3} = \frac{5}{3}$$

$$z_T = \frac{z_A + z_B + z_G}{3} = \frac{3+4+2}{3} = 3$$

5 bodova 7. Promatrajmo uspravni VALJAK takav da je njegov poprečni presjek kružnica polumjera 6 cm i visina 4 cm i opiši oko njega sferu. Dakle, gornja i donja osnovica valjka diraju sferu u svojim vanjskim obodima. Nadji obje površine ta dva dijela i njihov omjer: omjer površine (oplošja) sfere i površine valjka.



$$R = \sqrt{\left(\frac{v}{2}\right)^2 + r^2} = \sqrt{\left(\frac{4}{2}\right)^2 + 6^2}$$

$$R = \sqrt{16+36} = \sqrt{52}$$

$$P_{\ominus} = 4R^2\pi = 4 \times 52 \times \pi = \underline{204\pi} = 204 \times 3.14159\ldots$$

$$P_{\square} = 2B + P_{pl} = 2r^2\pi + 2r\pi h = 2r\pi(r+h) = 12\pi(6+4) = \underline{120\pi}$$

$$\frac{P_{\ominus}}{P_{\square}} = \frac{204}{120} = \frac{102}{60} = \frac{51}{30}$$

4 boda 8. Neka je $e = (e_1, e_2)$ baza vektorskog prostora $\mathbb{R}^2$ gdje je

$$e_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(u standardnoj bazi $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ od $\mathbb{R}^2$; $e_1 = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ itd.) i linearni operator $h : \mathbb{R}^2 \to \mathbb{R}^2$ dan vrijednostima na bazi, $h(e_1) = e_1 - e_2$, $h(e_2) = 2e_1 + e_2$.

a) koliko je $h(e_1 + e_2)$ u bazi e_1, e_2; (koristi linearnost od h!) ?

b) Rezultat napiši i u standardnoj bazi.

c) Napiši vektor $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ kao linearnu kombinaciju $\lambda e_1 + \mu e_2$ (nadji $\lambda, \mu \in \mathbf{R}$).

$$h(e_1+e_2) \underset{\substack{\uparrow \\ \text{linearnost od } h}}{=} h(e_1) + h(e_2) = e_1 - e_2 + 2e_1 + e_2 = \underbrace{3e_1}_{a)} = \underbrace{\begin{pmatrix} 3 \\ 6 \end{pmatrix}}_{b)}$$

c) $$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda + 3\mu \\ 2\lambda + \mu \end{pmatrix}$$

$\lambda + 3\mu = 4$

$2\lambda + \mu = 1 \Rightarrow \mu = 1 - 2\lambda$

$\lambda + 3(1-2\lambda) = 4 \Rightarrow -5\lambda = 1 \Rightarrow \lambda = -\frac{1}{5}$

$\mu = 1 - 2\left(-\frac{1}{5}\right) = \frac{7}{5}$

provjera:

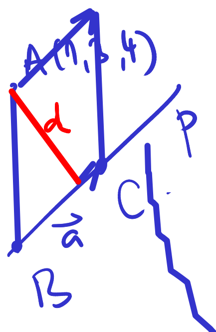
$$-\frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{7}{5}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} + \frac{21}{5} \\ -\frac{2}{5} + \frac{7}{5} \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

3 boda 9. Nadji kut izmedju ravnina $2x + 2y + 3z - 1 = 0$ i $3x + y + z - 1 = 0$ (priznaje se i ako nadjete samo kosinus ili sinus tog kuta).

to je ujedno kut između njihovih normala, a pripadni normalni vektori su (2,2,3), (3,1,1), dakle

$$\cos\theta = \frac{(2,2,3)\cdot(3,1,1)}{\sqrt{2^2+2^2+3^2}\,\sqrt{3^2+1^2+1^2}} = \frac{11}{\sqrt{17}\,\sqrt{11}} = \sqrt{\frac{11}{17}}$$

5 bodova 10. Nadji udaljenost od pravca $t \mapsto (-1+t, -2+t, 1-2t)$ do točke $A(1,3,-1)$.



$$p \ldots \vec{r}(t) = \underbrace{(-1,-2,1)}_{B} + t\,\underbrace{(1,1,-2)}_{\vec{a}}$$

$$d = \frac{\|\vec{a} \times \vec{BA}\|}{\|\vec{a}\|} \leftarrow \text{formula}$$

$$\vec{BA} = (1-(-1), 3-(-2), 4-1) = (2,5,3)$$

$B = (-1,-2,1)$, $A = (1,3,4)$

$$\vec{a} \times \vec{BA} = \begin{array}{ccc} + & - & + \\ \end{array}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 2 & 5 & 3 \end{vmatrix} = 13\vec{i} - 7\vec{j} + 3\vec{k}$$

$$\|\vec{a}\| = \|(1,1,-2)\|.$$

$$\|\vec{a}\| = \sqrt{1^2+1^2+(-2)^2} = \sqrt{6}$$

$$\|\vec{a} \times \vec{BA}\| = \sqrt{13^2+7^2+3^2} = \sqrt{169+49+9} = \sqrt{227}$$

$$d = \sqrt{\frac{227}{6}}$$