24 bodova (12 je 2, 15 je 3, 18 je 4 i 21 je 5). Upisivao sam 4 i za 17.5.

mat4 13.6.2023. IME i PREZIME:

2 boda

1. Čini li skup cijelih brojeva magmu s obzirom na dijeljenje ? Objasni zašto da ili ne.

Ne jer neki brojevi su djeljivi, a neki nisu, pa je operacija dijeljenja samo djelomično definirana, nije zatvorena (to je jedino svojstvo opće binarne algebarske strukture tj. magme koje treba provjeriti). Npr. 2 podijeljeno s 3 nije definirano.

3 boda

2. Dana je kvadratna matrica,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

Neka je $A^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ inverz to jest $AA^{-1} = I = A^{-1}A$ (množenje matrica daje jediničnu).

Nadji formulu za e u terminima a, b, c, d

dvije su matrice jednake ako imaju jednake sve elemente na istim mjestima

$$A^{-1} \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

daje u prvom retku jednadžbe

supstitucija

$$ea + fc = 1$$

$$eb + fd = 0 \Rightarrow f = -ebd^{-1}$$

$$ea - ebd^{-1}c = 1$$

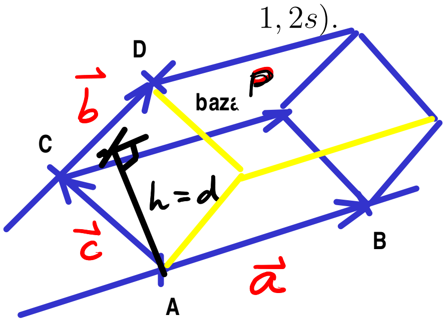
$$e(a - bd^{-1}c) = 1$$

$$\Rightarrow e = \frac{1}{a - bd^{-1}c} = (a - bd^{-1}c)^{-1} = \frac{1}{a - b/d \cdot c}$$

4 boda

3. Nadji udaljenost izmedju pravaca $t \mapsto (2t, t-1, t+1)$ i $s \mapsto (s-1, s+1, 2s)$.

$\vec{r}_1(t)$ $\qquad$ $\vec{r}_2(t)$



$d =$ visina paralelepipeda

$$P = \|\vec{a} \times \vec{b}\|$$

$$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}| = \pm \begin{array}{cl} \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 2 & -1 \end{vmatrix} & \begin{array}{l} \leftarrow \vec{a} \\ \leftarrow \vec{b} \\ \leftarrow \vec{c} \end{array} \end{array} = 8$$

$$\vec{r}_1(t) = \underset{A}{(0,-1,1)} + t\,\underset{\vec{a}}{(2,1,1)} = A + t\vec{a} \qquad \vec{c} = \vec{AC} = (-1,2,-1)$$

$$\vec{r}_2(t) = \underset{C}{(-1,1,0)} + s\,\underset{\vec{b}}{(1,1,2)} = C + s\vec{b}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\vec{k}$$

$$= \vec{i} - 3\vec{j} + \vec{k} \qquad \|\vec{a} \times \vec{b}\| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$$

$$V = P \cdot h = P \cdot d$$

$$d = \frac{V}{P} = \frac{8}{\sqrt{11}}$$

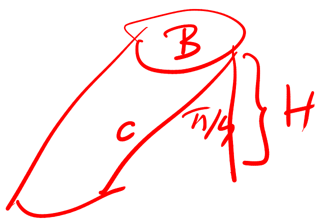
visina h
||
udaljenost d

Formula je obrazložena u videu o udaljenosti među paralelnim pravcima u prostoru

https://www.youtube.com/watch?v=reGftBYovL0

4. Promatrajmo kosi valjak takav da je kut izmedju osi valjka i okomice na osnovicu valjka $\pi/4$. Ako je volumen valjka 80 kubičnih centimetara i osnovica valjka je krug površine 20 kvadratnih centimetara, kolika je visina valjka H i kolika je duljina c svake izvodnice valjka (izvodnice spajaju ekvivalentne točke na rubu gornje i donje osnovice valjka).

3 boda



$$V = B \cdot H \Rightarrow H = \frac{V}{B} = \frac{80\,cm^3}{20\,cm^2} = 4\,cm$$

$$\frac{H}{c} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow c = \sqrt{2}\,H = \sqrt{2}\cdot 4\,cm \approx 5.656\,cm$$

5. Neka je $e = (e_1, e_2)$ baza od $\mathbb{R}^2$ dana vektorima

$$e_1 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} =$$

(u standardnoj bazi $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ od $\mathbb{R}^2$; $e_1 = 6\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ itd.)

Ako je linearni operator $h\mathbb{R}^2 \to \mathbb{R}^2$ dan vrijednostima na bazi, $h(e_1) = -4e_1 + 2e_2$, $h(e_2) = 3e_1 + 3e_2$.

1 bod a) koliko je $h(e_1 + 3e_2)$ u bazi e_1, e_2; (koristi linearnost od g!)

1 bod b) Rezultat napiši i u standardnoj bazi.

c) Napiši vektor $\vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ kao linearnu kombinaciju vektora baze e_1, e_2.

a) $h(e_1+3e_2) \underset{=}{=} h(e_1) + 3h(e_2) = -4e_1 + 2e_2 + 3(3e_1+3e_2) = 5e_1 + 11e_2$

zbog linearnosti od $\underline{\underline{h}}$

b) $= 5\begin{pmatrix} 6 \\ 2 \end{pmatrix} + 11\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 41 \\ 43 \end{pmatrix}$

c) $\lambda\vec{e}_1 + \mu\vec{e}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\quad \lambda\begin{pmatrix} 6 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$6\lambda + \mu = 4 \longrightarrow -8\mu = -5$

$2\lambda + 3\mu = 3 \Rightarrow -6\lambda - 9\mu = -9$

$6\lambda = 4 - \frac{5}{8}$

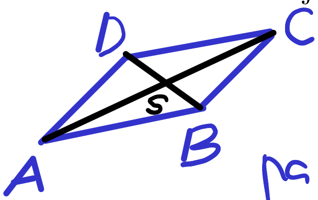
$6\lambda = \frac{27}{8}$

$\longrightarrow \lambda = \frac{27}{6\cdot 8} = \frac{9}{16}$

$\mu = \frac{5}{8} = \frac{10}{16}$

4 boda

6. Ako su tri vrha paralelograma u prostoru redom $A(2,1,3), B(3,4,2), C(1,0,1)$ nadji četvrti vrh D koji je nasuprotan vrhu B i sjecište dijagonala S.



$D = A + \overrightarrow{AD} = A + \overrightarrow{BC}$

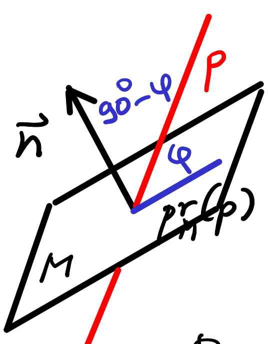
(ili $C + \overrightarrow{CD} = C + \overrightarrow{BA} = C - \overrightarrow{AB}$)

pa je $x_D = x_A + x_C - x_B \Rightarrow D(0,-3,2)$

$$\underbrace{x_S = x_A + \frac{1}{2}(x_C - x_A)}_{S = A + \overrightarrow{AS} = A + \frac{1}{2}\overrightarrow{AC}} = \frac{1}{2}(x_A + x_C) \qquad S\left(\frac{3}{2}, \frac{1}{2}, 2\right)$$

4 boda

7. Nadji kut izmedju ravnine $\overbrace{2x+y+z-2=0}^{\vec{n}(2,1,1)}$ i pravca p s parametrizacijom $t \mapsto (2t, t-1, -t+3)$ (priznaje se i dovoljno je i ako nadjete samo kosinus ili sinus tog kuta, ali naznači koji).



$$\varphi = \sphericalangle(M,p) = 90° - \overbrace{\sphericalangle(p, \underbrace{\vec{n}}_{\text{normala}})}^{90°-\varphi}$$

$$\| \quad \sphericalangle(\underbrace{pr_M(p)}_{\text{okomita projekcija } p \text{ na } M}, p)$$

$$\cos(90°-\varphi) = \underbrace{\frac{\vec{a}\cdot\vec{n}}{\|\vec{a}\|\cdot\|\vec{n}\|}}_{} = \frac{4}{\sqrt{6}\sqrt{6}} = \frac{4}{6} = \frac{2}{3}$$

$$p \ldots \vec{r}(t) = (0,-1,3) + t\underbrace{(2,1,-1)}_{\vec{a}}$$

$$\vec{a}\cdot\vec{n} = (2,1,-1)\cdot(2,1,1) = 2\cdot 2 + 1\cdot 1 - 1\cdot 1 = 4$$

$$\|\vec{a}\| = \sqrt{2^2+1^2+(-1)^2} = \sqrt{6} \qquad \|\vec{n}\| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$$

$$\sin\varphi = \cos(90°-\varphi) = \frac{2}{3}$$

Ako netko želi kosinus,

$$\cos\varphi = \sqrt{1-\sin^2\varphi} = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$$