Matematika 4, 8. travnja 2021.

modularna aritmetika = računske operacije za cijele brojeve modulo neki broj

UVIJEK dobivamo prsten $\underbrace{\mathbb{Z}/p\mathbb{Z}}_{\text{normalna podgrupa, za } +} = (\{[0],[1],\dots,[p-1]\}, +, \cdot, 0, 1)$

Taj prsten je polje ukoliko je p prosti broj ili potencija prostog broja na neki prirodni broj

KOMUTATIVNO TIJELO

$\forall r \neq 0, \exists r^{-1}, r^{-1} r = 1 = r \cdot r^{-1}$

Eckmann-Hiltonov argument: pretpostavimo da na nekom skupu imamo dvije operacije $\circ, *$ i vrijedi

$$(a * b) \circ (a' * b') = (a \circ a') * (b \circ b') \quad \forall a, a', b, b'$$

i s obzirom na obje imamo neutralni element 1

$$1 * a = a * 1 = 1$$
$$1 \circ a = a \circ 1 = 1$$

$a = 1, b' = 1$ $\quad (1 * b) \circ (a' * 1) = (1 \circ a') * (b \circ 1)$

$$b \circ a' = a' * b \Rightarrow \text{Abelova}$$

$b = 1, a' = 1$ $\quad \underline{\underline{a \circ b'}} = \underline{\underline{a * b'}} \Rightarrow \circ = *$

$\Rightarrow *$ i $\circ$ su iste i operacija je komutativna!

Polinomi, bis

polinomi s jednom varijablom i s koeficijentima u nekom komutativnom prstenu sami čine prsten

$$\mathbb{Q}[x] = \{a_n x^n + \dots + a_1 x^1 + a_0 \mid a_0, a_1, \dots, a_n \in \mathbb{Q}\}$$
$$\mathbb{Z}[x], \mathbb{C}[x]$$

OSNOVNI TEOREM ALGEBRE

$$\forall P \in \mathbb{C}[x], P \neq \text{const} \Rightarrow \exists c \in \mathbb{C}, \ P(c) = 0 \ (\text{nultočka})$$

TEOREM O DIJELJENJU POLINOMA S OSTATKOM

$m = n \cdot q + r$ (ostatak), $0 \leq r < n$

$$\forall P, S \ \exists R, Q \qquad P = \underset{\text{kvocijent}}{Q} \cdot S + \underbrace{R}_{\text{residue}} \ ; \ \deg R < \deg S$$

$$(3x^2+2x+5):(2x+1)=\frac{3}{2}x+\frac{1}{4}$$
$$\begin{array}{l} -(3x^2+\frac{3}{2}x) \\ \hline 0+\frac{1}{2}x+5 \\ -(\frac{1}{2}x+\frac{1}{4}) \\ \hline \frac{19}{4}=R \end{array}$$

P, S, Q

$$\underbrace{\left(\frac{3}{2}x+\frac{1}{4}\right)}_{Q}\cdot\underbrace{(2x+1)}_{S}+\underbrace{\frac{19}{4}}_{R}=3x^2+\frac{3}{2}x+\frac{\cancel{2}^{1}}{\cancel{4}_{2}}x+\frac{1}{4}+\frac{19}{4}$$
$$=3x^2+2x+5=P$$

EUKLIDOV ALGORITAM za nalaženje najveće zajedničke mjere dva polinoma, baš kao za prirodne brojeve

$25:10=2$
5
$10:5=2$
0

$M(25,10)=5$ $\quad V(25,10)=\frac{25\cdot10}{5}=50$

GCD greatest common divisor

djelitelj, mjera, nazivnik

najmanji zaj. višekratnik

$140, 72$

$140:72=1$
68

$72:68=1$
4

$68:4=17$
0

$M(140,72)=4$

polinomi nad poljem (da bi koef. djelili)

recimo $\mathbb{R}, \mathbb{Q}, \mathbb{C}$

P: x^2+2x+1, Q: x^2-1

$$(x^2+2x+1):(x^2-1)=1$$
$$\begin{array}{l} -(x^2 \quad -1) \\ \hline 2x+2 \end{array}$$

$$(x^2-1):\boxed{(2x+2)}=\frac{1}{2}x-\frac{1}{2}$$
$2(x+1)$
$$\begin{array}{l} -(x^2+x) \\ \hline -x-1 \\ -(-x-1) \\ \hline 0 \end{array}$$

$$M(\underbrace{x^2+2x+1}_{(x+1)^2}, \underbrace{x^2-1}_{(x+1)(x-1)})=x+1$$

Polinom je ireducibilan ili prost ako nije umnožak polinoma manjeg stupnja (ovisi nad kojim poljem gledamo)

Svaki polinom nad poljem možemo rastaviti na proste faktore na jedinstven način do na poredak i umnožak na invertibilnu konstantu
(kažemo da je prsten polinoma prsten s jedinstvenom faktorizacijom na proste faktore)

$$x+1=7\cdot\left(\frac{1}{7}x+\frac{1}{7}\right)$$

x^2+1 OSNOVNI TEOREM ALG. : ima nultočke nad $\mathbb{C}$

$x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x=\pm i$

$(x-(+i))(x-(-i)) = \underline{\underline{(x-i)}}\,\underline{\underline{(x+i)}}$

x^2+1 je rastavljiv (nije prost) nad $\mathbb{C}$

x^2+1 je prost nad $\mathbb{R}$, x^2+1 nema rastav nad $\mathbb{R}$

Ako je $\underbrace{c \text{ nultočka polinoma } P}$, onda je P djeljiv (bez ostatka) s (x-c)

Dokaz. Po teoremu o dijeljenju s ostatkom $P(x) = Q(x)\cdot(x\text{-}c) + R(x)$ $\underline{\text{uvrsti } c}$

$\cancel{P(c)}^{0} = Q(c)\cdot(\cancel{c-c}^{0}) + R(c)$ $\deg R < \deg(x-c)=1$, $\deg R = 0$

$\Rightarrow \quad 0 = R(c) \Rightarrow c$ je nultočka od R

$0 = R$

$P(x) = Q(x)\cdot(x-c)$.

$a_nx^n + \dots + a_0$, ima nultočku x_1 nad $\mathbb{C}$

$\underbrace{(a_nx^n+\dots+a_0)}_{n} : \underbrace{(x-x_1)}_{1} = \underline{b_{n-1}x^{n}+\dots+b_0}_{\,n-1}$ mult. x_2

$: (x-x_2) = \dots$ mult. x_3

$\vdots$

n nultočki nad $\mathbb{C}$

$(a_nx^n+\dots+a_0) : \overbrace{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}^{n}$

$= \underline{\underline{a_n}}$ unarni polinom $: a_n = 1$

$a_nx^n + \dots + a_0 = a_n(x-x_1)(x-x_2)\cdots(x-x_n)$

$x_1, x_2, \dots, x_n \in \mathbb{C}$ (neki mogu biti isti!)

$(x-1)^2$

$$\underline{x^2+px}+q=0$$

$$\left(x+\frac{p}{2}\right)^2 = x^2+2x\cdot\frac{p}{2}+\left(\frac{p}{2}\right)^2 = \underline{x^2+px}+\frac{p^2}{4}$$

$$\left(x+\frac{p}{2}\right)\left(x+\frac{p}{2}\right)$$

$$x^2+px+q = \underbrace{x^2+px+\frac{p^2}{4}} - \frac{p^2}{4} + q$$

$$\left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4} = 0$$

$$\left(x+\frac{p}{2}\right)^2 = -q+\frac{p^2}{4} \quad / \sqrt{\ }$$

$$x+\frac{p}{2} = \pm\sqrt{\frac{p^2}{4}-q} = \pm\sqrt{\frac{p^2-4q}{4}}$$

$$ax^2+bx+c=0$$

$$a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]=0$$

$$a\left[\left(x+\frac{b}{2a}\right)^2 \underbrace{-\frac{b^2}{4a^2}+\frac{c}{a}}\right]=0$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c}{a} = \frac{b^2-4ac}{4a^2} \quad / \sqrt{\ }$$

$$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \frac{\pm\sqrt{b^2-4a}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

$$= a\left(x^2-(x_1+x_2)x+x_1x_2\right)$$

$$= ax^2 - \underbrace{a(x_1+x_2)}_{-b}x + \underbrace{ax_1x_2}_{c}$$

VIETEOVE FORMULE

$$x_1\cdot x_2 = \frac{c}{a}$$

$$x_1+x_2 = -\frac{b}{a}$$

$$x_1 \cdot x_2 = \frac{c}{a} \qquad x_1 + x_2 = -\frac{b}{a}$$ VIETEove formule

Napiši neku kvadratnu jednadžbu s rješenjima 2 i 7

$$(x-2)(x-7) = 0$$

$$4(x^2 - 9x + 14) = 0$$

$$4x^2 - 36x + 56 = 0$$

$$2 \cdot 7 = \frac{c}{a} \qquad \underbrace{2+7}_{9} = -\frac{b}{a}$$

$$a = 4 \Rightarrow c = 2 \cdot 7 \cdot 4 = 56$$

$$b = -9 \cdot 4 = -36$$

Zamislimo sad slučaj kad su svi koeficijenti polinoma realni brojevi

$$\overline{x+iy} = x - iy$$

$$ax^2 + bx + c = 0 \quad , \quad a, b, c \in \mathbb{R}$$

$$a = \bar{a} \qquad b = \bar{b} \qquad c = \bar{c}$$

$$a(x - x_1)(x - x_2) = 0$$

$$x_1 + x_2 = -\frac{b}{a} \in \mathbb{R}$$

$$a' + b'i \qquad c' + d'i$$

$$Re\,x_1 + Im\,\cancel{x_1} \cdot i + Re\,x_2 + Im\,\cancel{x_2} \cdot i$$

$$Im\,x_1 = -Im\,x_2$$

$$b' = -d'$$

$$x_1 \cdot x_2 = \frac{c}{a} \in \mathbb{R}$$

$$(a' + b'i)(c' - b'i) = (a'c' + b'^2) + (b'\cancel{c'} - a'b')i \in \mathbb{R}$$

$$b'(c' - a') \to 0$$

dvije mogućnosti

1) $b' = 0 = -d'$ $\quad x_1, x_2 \in \mathbb{R}$ oba realna rješenja

2) $c' = a'$, $b' = d'$ $\quad \left.\begin{array}{l} x_1 = a' + b'i \\ x_2 = a' - b'i \end{array}\right\} x_1 = \bar{x}_2$

međusobno kompleksno konjugirana rješenja

KUBNI POLINOM S REALNIM KOEFICIJENTIMA

$$(a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0) : (x - x_1) = \underset{\in \mathbb{R}}{a_3}(x^2 + px + q)$$

$$a_3 (x - x_1)(x^2 + px + q)$$

$$(x - x_2)(x - x_3)$$

$$a_3(x-x_1)(x-x_2)(x-x_3)$$

$$a_3\big(x^3-\underbrace{(x_1+x_2+x_3)}_{-a_2/a_3}x^2+\underbrace{(x_1x_2+x_1x_3+x_2x_3)}_{a_1/a_3}x\underbrace{-x_1x_2x_3}_{-a_0/a_3}\big)$$

$$\underbrace{\cancel{x_1}+\cancel{x_2}}+x_3=-\frac{a_2}{a_3}\in\mathbb{R}$$

$$x_1x_2+x_1x_3+x_2x_3\in\mathbb{R}$$

$$x_1x_2x_3\in\mathbb{R}$$

jedna nultočka mora biti realna jer je polinom neparnog stupnja, a imaginarni dijelovi se moraju poništavati, druge dvije nultočke ili realne ili par kompleksno konjugiranih nultočki

polinom n-tog stupnja s realnim koeficijentima

ima nultočkekoje su ili realne ili u kompleksno konjugiranim parovima,

VIETE

$$\begin{cases} x_1+x_2+\dots+x_n=-\dfrac{a_{n-1}}{a_n}\\ x_1x_2+x_1x_3+\dots+x_{n-1}x_n=\dfrac{a_{n-2}}{a_n}\\ \vdots\\ x_1x_2x_3\cdots x_n=(-1)^n\dfrac{a_0}{a_n}\end{cases}$$

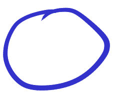
7-mi stupanj, $P(x)$, $P\in\mathbb{R}[x]$

Ima ili 7 realnih

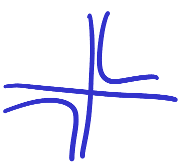
ili 5 realnih, 1 k.k. par

ili 3 realne, 2 k.k. para

ili 1 realnu, 3 k.k. para



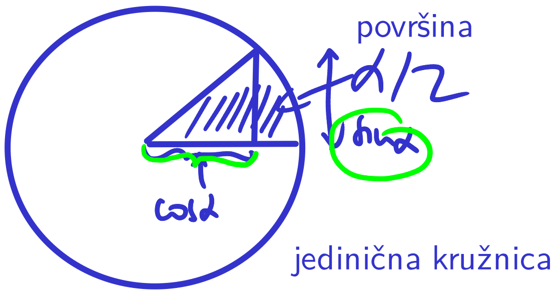
$x^2+y^2=1$ $\qquad x^2+y^2=(x+yi)(x-yi)$



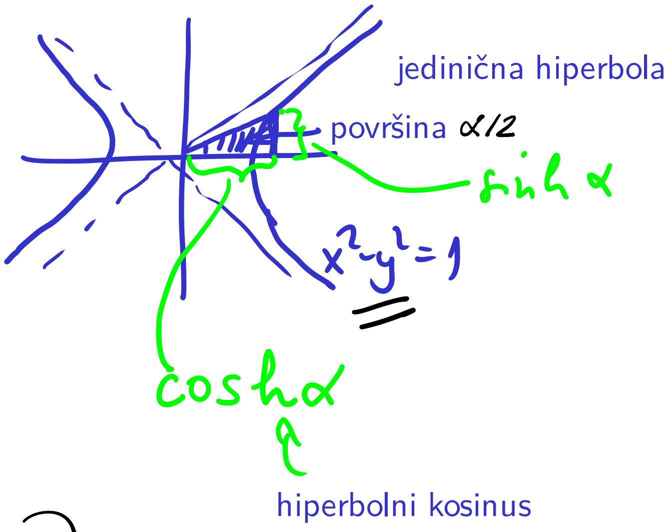
$x^2-y^2=1$ $\qquad x^2-y^2=(x-y)(x+y)$ $\quad \mathbb{R}$

$\boxed{x^2+1}=(x-i)(x+i)$ $\quad \mathbb{R}$

$x^2-1=(x-1)(x+1)$



$x^2+y^2=1$



Eulerova formula

$$\left.\begin{aligned} e^{i\varphi} &= \cos\varphi + i\sin\varphi \\ e^{-i\varphi} &= \cos(-\varphi) + i\sin(-\varphi) \\ e^{-i\varphi} &= \cos\varphi - i\sin\varphi \end{aligned}\right\} +$$

$$2\cos\varphi + \cancel{i\sin\varphi} - \cancel{i\sin\varphi} = e^{i\varphi} + e^{-i\varphi}$$

$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2} \qquad \sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i}$$

$$\cosh\varphi = \frac{e^{\varphi}+e^{-\varphi}}{2} \qquad \sinh\varphi = \frac{e^{\varphi}-e^{-\varphi}}{2}$$

$\mathrm{ch}\,\varphi$ $\qquad$ $\mathrm{sh}\,\varphi$

MORAL: kompleksni brojevi imaju geometrijski smisao, promjene slučajeva hiperbola/elipsa i slično sinh / sin ...